On Solution of Viscoelastic Fractional Differential Equation

by

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Abstract:

In this paper we study the fractional viscoelastic differential equation and solved it in general. We notice that our method is a generalization of Coimbra's approach and easier than Ayala method.

1- Introduction:

Fractional calculus is an old and new branch of mathematics with a long history. It's early beginning was in 1695 when G. W. Leibniz wrote a letter from Hanover, Germany, September 30, 1695 to G. A. L'Hopital said that $d^{\frac{1}{2}}x = x\sqrt{\frac{dx}{x}}$ which is an apparent paradox and this was found in volume 2, pp.301-302, Olms Verlag, Hildesheim, Germany 1962 and first published in 1849. After two years Leibniz wrote a letter to Wallis to discuss infinite product of π and in this letter Leibniz mentioned to differential calculus and used $d^{\frac{1}{2}}y$ to derivative of order $\frac{1}{2}$. For most details of historical background see [1,5].

The first application of fractional calculus is due to Abel in 1823 in solving an integral equation which arises in the tautochrone problem which is sometimes called isochrone problem and it is of finding the shape of a fractionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire slides to the lowest point of the wire in the same time regardless of where the bead is placed.

Many definitions of fractional derivatives and fractional integrals (fractionl differintegration for short) were introduced. These definitions agree when the order is integer and some of them are different when the order is not integer [4,10].

Although fractional calculus is old but it is not well known as that of integer order because a number of open questions were posted and one of them is the geometric and physical interpretation [7] gave an attempt to the solution of this open problem and interpret it as a shadow on the wall. So we saw the researchers did not depend on one definition of fractional differintegration but each one used the suitable definition for him and we think for our work that the definition of Nishimoto is applicable as in the following section.

2- Nishimoto Definition for Fractional Differintegration:

Through our study in complex analysis we use Cauchy integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz \quad ...(1)$$

where C is piecewise smooth Jordan curve in the complex plane, $i=\sqrt{-1}$ and z_0 lies inside C. In this formula we compute the complex integration by using the derivative. Nishimoto generalized this formula to compute the fractional derivative of a function of single variable as follows.

Definition(derivative): If *f(z)* is analytic function and it has no branch point on and inside *C* where $C=\{C_-, C_+\}$ and

$$C_- f^{(n)} = C_- f^{(n)}(z_o) = \frac{\Gamma(n+1)}{2\pi i}\int_{C_-} \frac{f(z)}{(z-z_0)^{n+1}}dz ; \Gamma \, is \, Gamma \, function ...(2)$$

$$= \frac{\Gamma(n+1)}{2\pi i}\int_{-\infty}^{0+} r^{-(n+1)} f(z_0+r)dr \qquad (3)$$

$$where \; r = z - z_0, z \neq z_0, -\pi \leq \arg(z-z_0) \leq \pi, and \; n \; is \; nonnegative \; integer$$

$$C_+ f^{(n)} = C_+ f^{(n)}(z_o) = \frac{\Gamma(n+1)}{2\pi i}\int_{C_+} \frac{f(z)}{(z-z_0)^{n+1}}dz \qquad ...(4)$$

$$= \frac{\Gamma(n+1)}{2\pi i}\int_{\infty}^{0+} r^{-(n+1)} f(z_0 + r)dr \qquad (5)$$

$$where\; r = z - z_0,\; 0 \le \arg(z - z_0) \le 2\pi, \quad n\; is\; non\; negative\; integer$$

$$f^{(-n)} = Cf^{(-n)} = \lim_{k \to -n} f^{(k)};\; n \in Z^{+} (+ve\; integer), C = \{C_{-}, C_{+}\} \ldots (6)$$

Where C_- and C_+ are the integral curves which are shown in fig.1 and fig.2, that is C_- is a curve along the cut joining two points z_0 and $-\infty + i\,\mathrm{Im}(z_0)$ and C_+ is a curve along the cut joining two points z_0 and $\infty + i\,\mathrm{Im}(z_0)$ then $f^{(n)} = Cf^{(n)}(z_0) = \{C_- f^{(n)}, C_+ f^{(n)}\}$ is the fractional derivative of order n of the function *f(z)* where $n \in R$ and $z \in C$ if $f^{(n)}$ exists.

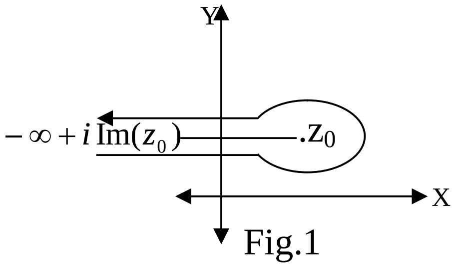


Fig.1

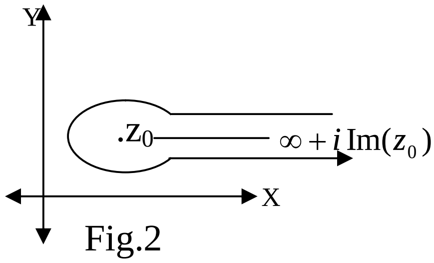


Fig.2

Definition(Integral): $f^{(n)}(n < 0)$ is the fractional integral of order $|n|$. That is the derivative of fractional order $-n$ ($n>0$) is the fractional integral of order n, ($n \in R$) if $f^{(n)}$ exists.

Notes:

1- If $f(z)$ is one valued analytic function on and inside $C = \{C_-, C_+\}$ then the fractional differintegration $f^{(n)}(z)$ of a function $f(z)$ can be defined in the above definition.

2- If $f(z)$ is analytic except at a number (finite or infinite) singular points not on and inside c then $f^{(n)}(z)$ can be defined again in the above definition.

3- $f(z)$ may be multible valued analytic function and in this case $f^{(n)}(z)$ is defined for principal value of $f(z)$.

4- If n is complex then we consider the principal value of n.

5- Z set of integers
Z^+ =N set of naturals
Z^- set of negative integers
R set of real numbers
C set of complex numbers.

3- Fractional differintegrations of some functions using Nishimoto definition

1- $\frac{d^r e^{ax}}{dx^r} = a^r e^{ax}$

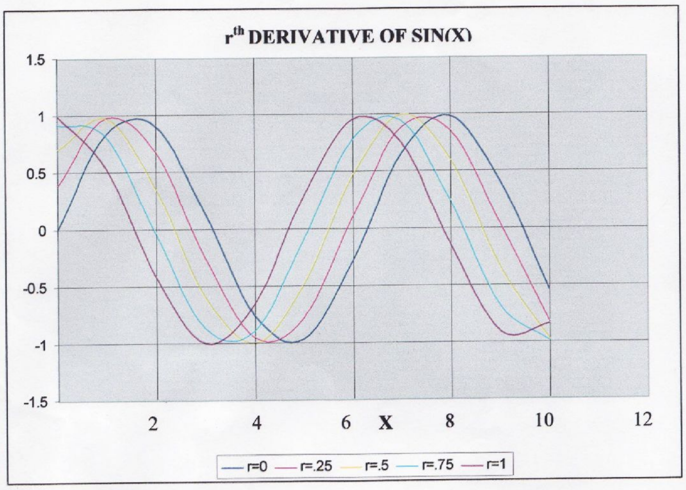
2- $\frac{d^r \sin(ax)}{dx^r} = a^r \sin(ax + \frac{\pi}{2} r)$

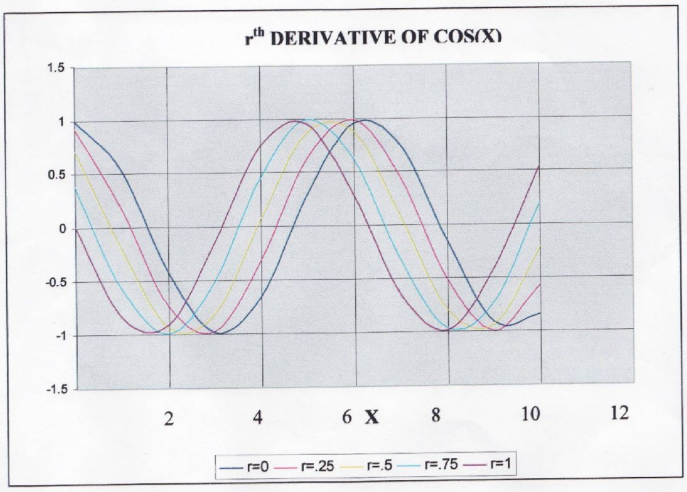
3- $\frac{d^r \cos(ax)}{dx^r} = a^r \cos(ax + \frac{\pi}{2} r)$

Where x is real or complex

In the following two figures we draw the rth derivative of *sinx* and *cosx*

3-5


r^th DERIVATIVE OF SIN(X)
1.5
1
0.5
0
-0.5
-1
-1.5
2
4
6
X
8
10
12
r=0
r=.25
r=.5
r=.75
r=1


r^th DERIVATIVE OF COS(X)
1.5
1
0.5
0
-0.5
-1
-1.5
2
4
6
X
8
10
12
r=0
r=.25
r=.5
r=.75
r=1

5

4- Derivation of Viscoelastic Oscillator [8,9]

Both Newton’s and d’Alembert’s method of developing equations of motion require that a mechanical system consisting of several components be taken apart and all forces acting on the components be identified on the free-body diagram of each element. However there are cases in which we have no interest in the forces at interconnections and it would be advantageous to be able to derive the equations of motion by regarding the system from an overall standpoint. J. L. Lagrange showed that a consideration of system energies allows such a derivation.

According to Lagrange’s approach a minimum number of independent coordinates necessary to describe the position of the system at any instant. Call these coordinates q_i and let Q_i be the corresponding loading in each coordinates. Let U be the potential energy of the system in terms of coordinates. When the system is displaced from its reference position then the potential energy includes energy stored in springs and the potential energy of weights raised above reference height. So the potential energy

$$U = f_1(q_i) \quad ...(7)$$

Where f_1 is a function depends on the problem under consideration. The Kinetic energy T in terms of the system masses, mass moments, linear velocities and angular velocities

$$T = f_2(\dot{q}_i^2) \quad ...(8)$$

The loss due to viscous fraction devices is written in an energy dissipation function that depends on the velocities of the system and the damping constants

$$R = f_3(\dot{q}_i^2) \quad ...(9)$$

Forces can be considered by including them in the force term Q_i associated with each coordinate. It can be shown by using d’Almbert principle that the equations of motion for each coordinate can be found by merely differentiati y function using the equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i \quad ...(10)$$

In spring mass damper $q_i = x$, therefore the energy functions are

$$U = \frac{1}{2}kx^2 \; , T = \frac{1}{2}m\dot{x}^2 \; and \; R = \frac{1}{2}b\dot{x}^2 \quad ...(11)$$

As in fig.4

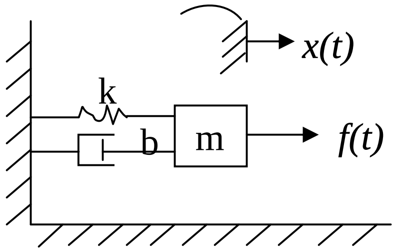


Fig.4 spring mass damper

Using (11) in (10) to get

$$m\ddot{x} + b\dot{x} + kx = f(t) \quad ...(12)$$

Where the force in the coordinate x is the excitation force $Q = f(t)$.

Studies had shown that a viscoelastic force is better expressed by a fractional derivative [3,9], therefore the new equation becomes

$$m\ddot{x} + bx^{(\frac{1}{2})} + kx = f(t) \quad ...(13)$$

5-The Solution of Function order Viscoelastic Equation [2,10]

The solution of equation (10) is the sum of the solution of the homogenous differential equation

$$m\ddot{x} + cx^{(\frac{1}{2})} + bx = 0 \quad ...(14)$$

and the particular solution of equation (10)

To solve equation (14) let the solution has the form $x = e^{at}$ then using in (14) we have

$$(ma^2 + ca^{\frac{1}{2}} + b)e^{at} = 0 \quad \sin ce \;\; e^{at} \neq 0 \;\; then$$

$$ma^2 + ca^{\frac{1}{2}} + b = 0$$

$$-ca^{\frac{1}{2}} = ma^2 + b$$

$$c^2 a = (ma^2 + b)^2 = m^2 a^4 + 2mba^2 + b^2$$

$$m^2 a^4 + 2mba^2 - c^2 a + b^2 = 0$$

$$i.e \quad a^4 + \frac{2b}{m}a^2 - \frac{c^2}{m^2}a + \frac{b^2}{m^2} = 0 \qquad \text{....(15)}$$

And we must find the value(s) of a satisfying this 4th order algebraic equation by using Ferrari's method.
The qubic equation to be solved is

$$a^3 + \frac{c^2}{m^2}a^2 - \frac{4b^2}{m^2}a - \frac{8b^3}{m^3} - \frac{b^4}{m^4} = 0 \quad \text{...(16)}$$

Let $a_2 = \frac{c^2}{m^2}, \; a_1 = -\frac{4b^2}{m^2} \; and \; a_0 = -\frac{8b^3}{m^3} - \frac{b^4}{m^4}$,then equation (16) becomes

$$a^3 + a_2 a^2 + a_1 a + a_0 = 0 \quad \text{...(17)}$$

To eliminate the term a^2 we put $a = y - \frac{1}{3}a_2$ then the equation (17) becomes

$$y^3 + py + q = 0 \quad \text{...(18)}$$

Where $p = a_1 - \frac{a_2^2}{3} \; and \; q = a_0 - \frac{a_1 a_2}{3} + \frac{2a_2^3}{27}$

And the roots of the (18) by using Cardans method are

$$y_1 = \sqrt[3]{A} + \sqrt[3]{B}, y_2 = w\sqrt[3]{A} + w^2\sqrt[3]{B} \; and \; y_3 = w^2\sqrt[3]{A} + w^3\sqrt[3]{B}$$

$$where \; A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \; B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, and \; w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Let y^* be any root of equation (18) then the solution of (15) are the solution of the two equations

$$\left.\begin{aligned} x^2+\frac{k}{2}x+\frac{1}{2}y^* &= ex+f \\ x^2+\frac{k}{2}x+\frac{1}{2}y^* &= -(ex+f) \end{aligned}\right\} \quad \ldots(19)$$

$$where\ e=\sqrt{\frac{k^2}{4}-b+y^*}\ and\ f=\sqrt{-d+\frac{1}{4}y^{*2}}$$

$$for\ x^4+kx^3+bx^2+cx+d=0$$

After finding the four roots of equation (17) we can compute the four roots of (15) and thus we find the complementary solution of equation (14). To find the particular solution $x_p(t)$ of equation (10) we depend on $f(t)$ in equation (10) as follows (Nishimoto)

1- $f(t)=Ae^{at}$ *we let* $xp(t)=Ae^{at}$ *and we find A.*
2- $f(t)=\cos(at)\,or\,\sin(at)$ *we let* $xp(t)=A\cos(at)+B\sin(at)\ ,a\neq 0$
3- $f(t)$ is any function we let $x_p(t)$ as a polynomial of certain degree.

Conclusions

1- In equation (1) we can replace the term $cx^{(\frac{1}{2})}$ *by* cx^q where q is any positive real number and in this case Coimbra's approach is a special case $(m=c=b=1)$.
2- This method is easier than that used by Ayala.
3- We can checkthe stability according to the real parts of the roots of equation (15) and also we can study oscillation of solution of this equation.

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