*Strongly-Pseudo-Extending Modules and SP-modules**

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Abstract

Let R be a ring and M be an R-module. Recall that M is extending if, every submodule of M is essential in a direct summand of M. And recall that an R-module M is fully pseudo stable if every submodule of M is pseudo stable.

In this work, we introduce and study two classes of modules. The first class is stronger than extending modules, and the second class is generalization of fully pseudo stable modules. We call an R-module M is strongly-pseudo-extending if, every submodule of M is essential in a pseudo stable direct summand of M. We call an R-module of M is SP-module if, every direct summand of M is pseudo stable. Many characterizations and properties of these concepts are given. Moreover, the relation among these concepts is studied. It is shown that an R-module M is strongly-pseudo-extending if and only if M is an extending and M is SP-module.

Introduction

Through out this paper, R will be denoted an associative commutative ring with identity, and all R-modules are unitary (left) R-modules.

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An R-module M is called extending if every submodule of M is essential in a direct summand of M. Extending modules have been studied recently by several authors, among them M. Harada, B. Muller, P.F. Smith, and J. Clark [3].

In this work, we introduce and study in section one the concept of strongly-pseudo-extending module which is stronger property than extending module.

An R-module M is called strongly-pseudo-extending if, every submodule is essential in a pseudo stable direct summand of M. A non-zero submodule N of an R-module M is called pseudo stable if for each R-monomorphism $f: N \to M, f(N) \subseteq N$ *[1]. And a non-zero submodule K of an R-module M is called essential in M, if* $K \cap L \neq (0)$ *for every non-zero submodule L of M [5].*

Several characterizations of strongly-pseudo-extending modules are given. Moreover, we investigate direct decomposition for strongly-pseudo-extending modules. Also inherited property for strongly-pseudo-extending modules is studied. We show that a closed (and hence direct summand) submodules of strongly-pseudo-extending module are strongly-pseudo-extending.

In section two of this paper, as a proper generalization of fully-pseudo stable modules and as a link between extending modules, and strongly-pseudo extending modules, we introduce, and study the concept SP-module. An R-module is called Sp-module, if every direct summand of M is pseudo stable. Many examples, properties and characterizations of this concept are given; we assert that extending modules and strongly-pseudo-extending modules are linked by SP-module. Known modules related to SP-module are considered. A direct summand of SP-module is SP-module.

§ 1: Strongly-Pseudo-Extending Modules

In this section, we introduce and study a class of modules which is stronger property than extending modules.

Definition 1.1

An R-module M is called strongly-pseudo extending if, every submodule of M is essential in a pseudo stable direct summand of M.

A ring R is called strongly-pseudo-extending if R is a strongly-pseudo-extending left (right) R-module.

Example and Remarks 1.2

1- *Every strongly-pseudo-extending module is extending, but the converse is not true in general.*
   For example $M = Z_{p^\infty} \bigoplus Z_{p^\infty}$ *is extending Z-module because M is injective-Z-module, while* $M = Z_{p^\infty} \bigoplus Z_{p^\infty}$ *is not strongly-pseudo-extending.*
2- *Every uniform module is strongly-pseudo-extending, but the converse is not true. For example,* Z_6 *as a Z-module is strongly-pseudo-extending but it is not uniform.*
3- *Every semi-simple fully-pseudo-stable module is strongly-pseudo-extending. But the converse is not true. For example the Z-module Q is strongly-pseudo-extending, but it is not fully pseudo- stable, and also by uniformity of* Q_Z *it is not semi-simple.*

The proof of the following proposition is straightforward and hence omitted.

Proposition 1.3

Let M be an R-module. Then M is uniform if and only if M is indecomposable and strongly-pseudo-extending.

The following result shows that the two concepts strongly-pseudo-extending modules and extending modules are equivalents in the class of indecomposable modules.

Proposition 1.4

Let M be an indecomposable module. Then M is strongly-pseudo-extending if and only if M is extending.

Proof: Obvious.

Recall that a submodule N of an R-module M is closed in M, if it has no proper essential extension in M [5].

In the following theorem we give many characterizations of strongly-pseudo-extending modules.

Theorem 1.5

Let M be an R-module. Then the following statements are equivalent.

1. *M is strongly-pseudo-extending module.*
2. *Every closed submodule of M is pseudo stable direct summand.*
3. *If A is a direct summand of $E(M)$, then $A \cap M$ is pseudo stable direct summand of M.*

<u>*Proof:*</u> $(1) \Longrightarrow (2)$

Let A be a closed submodule of M. Since M is strongly-pseudo-extending, then there exists a pseudo stable direct summand B of M such that A is essential in B. But A is a closed submodule of M, hence A=B. That is A is a pseudo stable direct summand of M.

$(2) \Longrightarrow (3)$. *Let A be a direct summand of* $E(M)$, *then* $E(M) = A \oplus B$, *where B is a submodule of* $E(M)$ *to show that* $A \cap M$ *is closed in M. Suppose that* $A \cap M$ *is essential in K, where K is a submodule of M, and let* $k \in K$. *Thus* $k = a + b$ *where* $a \in A$ *and* $b \in B$. *Now consider that* $k \notin A$, *then,* $b \neq 0$. *But M is essential in* $E(M)$ *and* $0 \neq b \in B \subseteq E(M)$, *therefore there exists* $r \in R$ *such that* $0 \neq rb \in M$. *Now,* $rk = ra + rb$ *and hence*

$ra = rb - rk \in M \cap A \subseteq K$. Thus, $rb = rk - ra \in K \cap B$. But $A \cap M$ is essential in K, so $(0) = (A \cap M) \cap B$ is essential in $K \cap B$ and hence $K \cap B = (0)$. Then $rb = 0$ which is a contradiction. Thus $A \cap M$ is closed in M and hence $A \cap M$ is a pseudo stable direct summand of M.

$(3) \Longrightarrow (1)$ Let A be a submodule of M. Let B be a relative complement of A, then $A \oplus B$ is essential in M [5]. But M is essential in $E(M)$, thus $A \oplus B$ is essential in $E(M)$ [5] and so $E(M) = E(A) \oplus E(B)$. Since $E(A))$ is a summand of $E(M)$, then by using (3) $E(A) \cap M$ is a pseudo stable direct summand of M. But A is essential in $E(A)$, and M is essential in M, then $A = A \cap M$ is essential in $E(A) \cap M$ [5]. Therefore M is strongly-pseudo-extending module. Θ

The decomposition theory for any algebraic structure has always a useful tool in the study of its properties and structure theory. The following result gives a decomposition theorem for strongly-pseudo-extending module.

Theorem 1.6

An R-module M is strongly-pseudo-extending if and only if for each submodule A of M, there is a direct decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$ where $M_1$is pseudo stable submodule of M and $A \oplus M_2$ is an essential submodule of M.

<u>Proof:</u>

Suppose that M is strongly-pseudo-extending module. Let A be a submodule of M. thus A is essential in a pseudo stable direct summand say K of M. That is $M = K \oplus K_1$ where K_1 is a submodule of M. Also, since A is essential in K and K_1 is essential in K_1 , thus $A + K_1$ is essential in $K \oplus K_1 = M$ [5]. Hence $A + K_1$ is essential submodule of M.

Conversely: let A be a submodule of M. By hypothesis, there is a direct decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$, where M_1 is a pseudo-stable submodule of M and $A + M_2$ is essential in M. We claim that A is essential submodule of M_1. Let K be a non-zero submodule of M_1, hence K is a submodule of M. Since $A + M_2$ is essential in M, then $(A + M_2) \cap K \neq (0)$. Let $k = a + m_2 \neq 0$, where $k \in K, a \in A$ and $m_2 \in M_2$, thus $m_2 = k.a$ which implies that $m_2 \in M_1 \cap M_2 = (0)$, therefore $k = a \in K \cap A \neq (0)$, then $K \cap A \neq (0)$, hence A is essential in M_1. Thus M is strongly-pseudo- extending module. Θ

Recall that a submodule N of an R-module M is fully invariant submodule if $f(N) \subseteq N$ for each $f \in End(M)$. [13]

Proposition 1.7

Fully invariant direct summands submodules are pseudo stable.

<u>Proof:</u>

Let K be fully invariant direct summands submodule of an R-module M. let $f: K \to M$ be any R-monomorphism. Since K is a direct summand of M, thus there is the projective mapping $\pi: M \to K$. Hence $f \circ \pi: M \to M$ is an R-homomorphism. Since K is fully invariant submodule of M, then we have $(f \circ \pi)(K) \subseteq K$, and so $f(K) = f(\pi(K) \subseteq K$. Thus K is a pseudo- stable submodule of M.

In proposition 1.7 if a submodule K of an R-module M is either fully invariant or direct summand but not both, then K need not be pseudo-stable submodule. For example in a Z-module Z the submodule 2Z is fully invariant, not pseudo-stable and it is not direct summand. Θ

Recall that an R-module M is called duo module if every submodule of M is fully invariant submodule of M [7].

The following corollaries are immediate consequences of Prop.1.7.

Corollary 1.8

Every duo semi-simple R-module is fully pseudo-stable.

An R-module M is called multiplication module if every submodule of M is of the form AM for some ideal A of R [2]

Corollary 1.9

Every multiplication semi-simple R-module is fully-pseudo-stable R-module.

It is well known that every cyclic R-module is multiplication, we have the following result.

Corollary 1.10

Every cyclic semi-simple R-module is fully-pseudo stable.

By using Prop.1.7 and definition of strongly-pseudo-extending module, we have the following result.

Proposition 1.11

If every submodule of an R-module M is essential in a fully invariant direct summand of M, then M is strongly-pseudo-extending.

Recall that an R-module M is quasi-injective, if each R-homomorphism $f:N \to M$ form any submodule N of M into M can be extended to an R-homomorphism of M [6].

It is well-known that every quasi-injective module is extending module [8], and we have every strongly-pseudo-extending module is extending.

A question arises about the relationship between quasi-injective modules and strongly-pseudo-extending modules. In fact they are independent concepts. The Z-module Z is strongly-pseudo-extending (since it is uniform) but it is not quasi-injective. On other hand the vector *space of dimension two over a filed F is quasi-injective module, while it is not strongly-pseudo-extending F-module.*

In the following results we consider conditions under which quasi-injective module is strongly-pseudo-extending.

Proposition 1.12

Every multiplication quasi-injective module is strongly-pseudo-extending.

Proof:

Suppose that M is multiplication quasi-injective R-module. Let K be a closed submodule of M. Since M is quasi-injective, then K is a direct summand of M [9, Lemma2]. It is enough to show that K is fully invariant submodule of M. Let $f \in End(M)$. Since M is a multiplication. Then $K = AM$ for some ideal A of R. Now, $f(K) = f(AM) = Af(M) \subseteq AM = K$. Hence K is fully-invariant. Therefore, by Prop. 1.7 K is a pseudo-stable submodule of M. Hence by Theorem 1.5 M is strongly-pseudo-extending. Θ

As an immediate consequence of Prop. 1.12 we get the following corollary.

Corollary 1.13

Every cyclic quasi-injective R-module is strongly-pseudo-extending.

Before we introduce the next result, we recall the following definitions.

Let R be a ring and H, N are submodules of an R-module M, the residual of H by N is $[H:N] = \{x \in R: xN \subseteq H\}$ and the annihilator of M denoted by $ann(M) = [0:M]$. Also recall that an R-module M is faithful if $ann(M) = 0$ [11].

If R is strongly-pseudo-extending ring, then M may not be strongly-pseudo-extending R-module. For example, consider the Z-module $Z \oplus Z_2$, *we observed that Z is strongly-pseudo-extending ring(since it is uniform), while* $Z \oplus Z_2$ *is not strongly-pseudo-extending Z-module.*

The following proposition gives a condition under which a module over extending ring is strongly-pseudo-extending

Proposition 1.14

Let M be a faithful multiplication module. If R is extending ring, then M is strongly-pseudo-extending.

Proof:

Let K be a closed submodule of M. Since M is a multiplication, then $K = [K:M]M$ *[4]. But K is closed in M, therefore by [7, Prop.(3.31)]* $[K:M]$ *is closed in R. Now, since R is extending ring, thus* $R = [K:M] \oplus J$ *where J is an ideal of R, and hence* $M = RM = ([K:M] \oplus J)M = [K:M]M + JM$. *Since M is faithful multiplication R-module, then by [4, Theorem 1.6]*

$[K:M]M \cap JM = ([K:M] \cap J)M = (0)$, *so*
$M = [K:M]\, M \oplus JM = K \oplus JM$.

That is K is a direct summand of M. To prove that K is fully invariant direct summand. Let $f \in End(M), and\ k \in K = [K:M]M$. *Then* $k = \sum_{i=1}^{l} s_i m_i$ *where* $s_i \in [K:M]\ and\ m_i \in M$. *Then,* $f(k) = f(\sum_{i=1}^{l} s_i m_i) = \sum_{i=1}^{l} s_i f(m_i) \in [K:M]M = K$. *Hence* $f(K) \subseteq K$ *that is K is a fully invariant submodule of M. Therefore K is a pseudo stable by Prop. 1.7 . Hence M is strongly-pseudo-extending by Theorem 1.5.Θ*

We don't know in general whether strongly-pseudo-extending property is inherited by submodules. The following results are partial answering.

Proposition 1.15

A closed submodule of strongly-pseudo-extending module is strongly-pseudo-extending.

Proof:

Let L be closed submodule of strongly-pseudo-extending R-module M. Let K be a closed submodule of L, then K is closed submodule of M [13, p18]. Since M is strongly-pseudo-extending, then K is a pseudo-stable direct summand of M by Prop.1.5. And since $K \subseteq L$ and K is a direct summand of M, then K is direct summand of L[10,Lemma 2.4.3]. To prove that K is a pseudo stable submodule of L. Let $f: K \to L$ be any R-monomorphism and consider $K \xrightarrow{f} L \xrightarrow{inc} M$, where(inc) is the inclusion mapping. Then $(inc \circ f): K \to M$ is an R-monomorphism. Since K is a pseudo stable submodule of M, then $(inc \circ f)(K) \subseteq K$. That is $f(K) \subseteq K$. Therefore K is a pseudo stable direct summand of L. Hence L is strongly-pseudo-extending. Θ

It is well known that every direct summand is closed we get the following result.

Corollary 1.16

A direct summand of strongly-pseudo-extending R-module is strongly-pseudo-extending.

Proposition 1.17

Let M b a strongly-pseudo-extending R-module, such that the intersection of every submodule N with any pseudo stable direct *summand of M is pseudo stable direct summand of N. Then N is strongly-pseudo-extending.*

Proof:

Let B be a submodule of N. Since M is strongly-pseudo-extending, and B is a submodule of M, then there exists a pseudo stable direct summand K of M such that B is essential in K. But $B \subseteq K \cap N \subseteq K$ *is essential in* $K \cap N$ *[13]. But by hypothesis* $K \cap N$ *is a pseudo stable direct summand of N. hence N is strongly-pseudo-extending.*

As we mention in Examples and Remark 1.2(1) that every strongly-pseudo-extending module is extending, but the converse is not true. In the following theorem we give a weaker condition to prove that the converse is true.

Theorem 1.18

Let M be an R-module such that every direct summand of M is pseudo stable, then M is strongly-pseudo-extending if and only if M is extending.

Proof: *obvious.*

§2: SP Modules

Recall that an R-module M is fully – pseudo stable of M if, every submodule of M is pseudo stable [1]. We note that in section one (Theorem 1.18) the concepts of strongly-pseudo-extending modules and extending modules are equivalent under the condition "every direct summand is a pseudo stable" This lead us to introduce and study this condition as a proper generalization of fully-pseudo stable modules as follows.

Definition 2.1

An R-module M is called SP-module if, every direct summand of M is pseudo stable.

As, we call a ring SP-ring if, R is SP-module as R-module.

Examples and Remarks

1. *Every uniform module is SP-module.*
2. *Q as a Z-module is SP-module, but not fully pseudo stable.*
3. *Every fully pseudo stable module is SP-module, but the converse is not true. For example the Z-module Z is SP-module, but not fully-pseudo stable.*
4. *From Prop. 1.7, if every direct summand of an R-module M is fully-invariant, then M is SP-module.*
5. *Every duo module is SP-module.*
6. *Every indecomposable module is SP-module.*
7. *Every strongly-pseudo-extending module is SP-module.*

We restate Theorem 1.18 as follows

Theorem 2.3

If an R-module M is SP-module, then M is strongly-pseudo-extending if and only if M is Extending.

Theorem 2.4

An R-module M is strongly-pseudo-extending if and only if M is extending and SP-module.

As an immediate consequence of Th.2.3, we have the following corollary.

Corollary 2.5

If M is a semi-simple R-module, then M is strongly-pseudo- extending if and only if M is SP-module.

The following proposition gives a characterization of SP-module in the class of extending modules.

Proposition 2.6

If M is extending module, then M is SP-module if and only if every closed submodule of M is pseudo stable.

Proof:

$(\Longrightarrow)$ *By theorem 2.4 M is strongly-pseudo extending module and by theorem 1.5 every closed submodule of M is pseudo stable.*

$(\Longleftarrow)$ *It is obvious.* Θ

Proposition 2.7

Let M be an R-module, such that each direct summand of M has a unique complement H such that $M = D \oplus H$, *then M is SP-module.*

Proof:

Let D be a direct summand of M, then there is a submodule C of M such that $M = D \oplus C$ *and consider* $\pi_C \, and \, \pi_D$ *the projection mappings of M onto D and C respectively. Assume that D is not pseudo stable submodule of M, then there exists an R-monomorphism* $f: D \to M$ *with* $f(D) \nsubseteq D$. *Moreover, we may extend f to M by putting* $f(x) = 0$ *for all* $x \in D$. *Then* $f \circ \pi_D = f \; and \; f \circ \pi_C = 0$. *Consider the two R-homomorphism's,*

$(\pi_C + \pi_C \circ f) \; and \; (\pi_D - \pi_C \circ f)$.

It is clear that $(\pi_C + \pi_C \circ f) + (\pi_D - \pi_C \circ f) = I$ *and* $(\pi_C + \pi_C \circ f) \circ (\pi_D - \pi_C \circ f) = (\pi_D - \pi_C \circ f) \circ (\pi_C + \pi_C \circ f) = 0$

That is $(\pi_C + \pi_C \circ f)$ *and*$(\pi_D - \pi_C \circ f)$ *are sum-1-orthogonal idempotent, therefore by [12, lemma 4.6] M is a direct sum of the submodules* $(\pi_C + \pi_C \circ f)(M)$ *and* $(\pi_D - \pi_C \circ f)(M) \nsubseteq D$. *Thus*

$M = C \oplus D'$ *where* $D' = (\pi_D - \pi_C \circ f)(M)$ *and* $D \neq D'$ *which is a contradictions with the assumption. Then D is pseudo stable submodule of M, and hence M is SP-module.* Θ

Proposition 2.8

Let M be an R-module, such that every decomposition $M = H \oplus K$ *(where H and K are submodule of M) with* $Hom_R(H, K) = 0$*. Then M is SP-module.*

<u>Proof:</u>

Let H be a direct summand of M, thus there is a direct summand K of M such that $M = H \oplus K$*. By hypothesis ,*$Hom_R(H, K) = 0$ *and by [7] H is a fully invariant submodule of M. Therefore H is a pseudo stable by Example and Remark 2.1 (4).* Θ

Before we give the next proposition, we introduce the following lemma.

Lemma 2.9

Every cyclic submodule of an R-module M is pseudo stable, then M is a fully-pseudo stable.

We noticed that every fully-pseudo stable module is SP-module, and the converse is not true in general. In the following proposition, we obtain a condition under which the converse is not true.

Firstly, recall that an R-module M is regular, if given any element m in M, there exists $f \in Hom_R(M, R)$ *such that* $m = f(m)m$ *[14].*

Proposition 2.10

Every regular SP-module is fully-pseudo stable

<u>Proof:</u>

Let N be any cyclic submodule of regular SP-module. By regularity of M, N is a direct summand of M [14, Th.2.2]. So, since M is SP-module, thus N is a pseudo stable submodule of M. Therefore by lemma 2.9 M is fully-pseudo stable. Θ

Corollary 2.11

If M is regular R-module then, M is fully-pseudo stable if and only if M is SP-module.

The condition of regularity of the module in corollary 2.11 is necessary because the Z-module Q of rational numbers is not regular and Q_Z *is SP-module but it is not fully-pseudo stable.*

The following result asserts that if $End_R(M)$ *is commutative it is sufficient to make M is SP-module.*

Proposition 2.12

Let M be an R-module such that $End_R(M)$ *is commutative. Then M is SP-module.*

<u>Proof:</u>

Let N be a direct summand of M and $f: N \to M$ *be any R-monomorphism. There exists a submodule K of M such that* $M = N \oplus K$*. Then f can be extended to an R-homomorphism* $g: M \to M$ *by putting* $g(k) = 0$ *for each k in K. Define* $h: M \to M$ *by* $h(x, y) = x$ *for each x in N and y in K. Let* $f(x) = y + z$ *for some y in N and z in K.*
Now, $(h \circ g)(w) = (h \circ g)(x + y) = h(f(x) = h(y + z) = y$ *,and on other hand* $(g \circ h)(w) = (g \circ h)(x + y) = g(x) = y + z$*. Since* $End_R(M)$ *is commutative, then* $h \circ g = g \circ h$*, and so* $z = 0$*. Then* $f(x) \in N$*, therefore* $f(N) \subseteq N$*, hence M is SP-module.* Θ

In the following Proposition, we prove that the class of multiplication modules is contained in the class of SP-modules.

Proposition 2.13

Every multiplication R-module is SP-module.

Proof:

Let N be a direct summand of a multiplication module M, and let $f: N \to M$ *be any R- monomorphism. Since M is multiplication, then N=AM for some ideal A of R. But N is a direct summand of M, thus f can be extended to an R-homomorphism* $g: M \to M$*. Now,* $f(N) = g(N) = g(AM) \subseteq AM = N$*. Thus N is a pseudo stable of M. Therefore M is SP-module. Θ*

The converse of Th. 2.13 is not true in general. For example the Z-module Q is SP-module, but not multiplication.

The following proposition shows that the direct summands of SP-module inherit the property.

Proposition 2.14

Every direct summand of SP-module is SP-module.

Proof:

Let M be SP-module and let N be a direct summand of M, and let $f: K \to N$ *be any R-monomorphism. Now, since N is a direct summand of M, then K is a direct summand of M [10]. Since M is SP-module, then K is pseudo stable submodule of M. thus* $i \circ f: K \to M, where\ i: N \to M$ *is the inclusion mapping, and so* $(i \circ f)(K) \subseteq K$*. That is* $f(K) \subseteq K$*. Thus K is a pseudo stable submodule of N. Hence N is SP-module.*

Remark 2.15

The direct sum of SP-modules needs not to be SP-module. For example, consider Z and Z_p as a Z-modules (where p is prime number). Since Z and Z_p are uniform Z-modules, then they are SP-modules. But $Z \oplus Z_p$ as Z-module is not SP-module. In fact, by uniformity of Z and Z_p the only direct summands of M are $\bar{0} \oplus \bar{0}$, $Z \oplus \bar{0}$, $\bar{0} \oplus Z$ and M. But $Z \oplus \bar{0}$ is not pseudo stable submodule of M. For if, defining $f: Z \oplus \bar{0} \to M$ by $f(x, \bar{0}) = (0, \bar{x})$ for each $(x, \bar{0}) \in Z \oplus \bar{0}$. Clearly f is a Z- monomorphism. But $f(Z \oplus \bar{0}) \nsubseteq Z \oplus \bar{0}$. Θ

Recall that an R-module M is a directly finite if M is not isomorphic to proper direct summand of itself [5,p165].

The following proposition shows that the class of SP-modules is contained in the class of directly finite modules.

Proposition 2.16

Every SP-module is directly finite.

<u>Proof:</u>

Let M be an SP-module. Suppose that $M \cong K$ where K is a proper direct summand of M. let y be a non-zero element in M which is not in K, and let $f: M \to K$ be an isomorphism. Consider the monomorphism, $f^{-1}: K \to M$ and $(i_K \circ f): M \to M$, where i_K is the inclusion mapping from K into M. Since M is SP-module, thus K is a pseudo-stable submodule of M, and $f^{-1}(K) \subseteq K$ and $(i_K \circ f)(M) \subseteq M$. Now $y = (f^{-1} \circ i_K \circ f)(y) \in K$ this is a contradiction.

Remark 2.17

The converse of Prop. 2.16 is not true in general, for example the two dimensional vector space V over a field F is directly finite (since it is finite dimension [5]). But V is not SP-module.

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مقاسات التوسع- الكاذبة- القوية والمقاسات من النمط-SP

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الخلاصة

لتكنR حلقة و M مقاساً. نقول ان M مقاس توسيع اذا كان كل مقاس جزئي من M يكون مركبة جمع مباشر من M و نقول ان المقاس M تام الاستقرارية كاذب اذا كان كل مقاس جزئي منه يكون مستقر كاذب.

في هذا العمل تم عرض ودراسة صنفين من المقاسات اولهما صنف اقوى من المقاس الموسع وثانيهما هو تعميم الى المقاس التام الاستقرارية الكاذب. حيث نقول ان المقاس M موسع – كاذب- قوي اذا كان كل مقاس جزئي منه يكون جوهري من مركبة جمع مباشر مستقر كاذب ، من جهة اخرى نقول ان المقاس M انه من النمط –SP اذا كان كل مركب جمع مباشر من M يكون مستقر كاذب.

تم اعطاء العديد من المكافئات و الخواص لهذين المفهومين . فضلا عن ذلك تم دراسة العلاقة بين هذين المفهومين. حيث برهنا ان المقاس M يكون موسع كاذب-قوي اذا وفقط اذا كانM مقاس موسع و M مقاس من النمط-SP .