# *Convergence of Riesz Fractional Integral*

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## Abstract

In this article convergence of Riesz fractional integral is studied by comparing the integrand $|x-t| f(t)$ where $0<n<1$ with certain functions. Variation of coordinates as well as complex analysis are used to prove the convergence. This enables us to use the comparison test of convergence widely.

## Introduction

Fractional calculus is commonly called generalized differintegration which means arbitrary orders (real, complex) derivatives and integrals [5]. Its history can dated back to 1695 when L.Hospital and Leibniz were communicating whether it made sence to define $\frac{d^n}{dx^n}$ for $n=\frac{1}{2}$ [2,6].

One can state that the whole theory of fractional derivatives and integrals was established in the 2nd half of the 19th century [2].

More than twelve definitions were given for fractional derivatives and integration [5,9].

Fractional calculus does not developed as that of integer order because of its lack of geometric and physical interpretation until 2001 when Podlubny [7] gave an interpretation for Riemann-Liouville fractional derivative and integration.

M. Riesz fractional integral of order n where 0<n<1 is defined by

$$f_n(x) = \int_{-\infty}^{\infty} |x - t|^n \; f(t)\, dt$$

In 1949 Riesz had developed a theory of fractional integration for functions of several variables [6].

Bassam 1951 in his Ph.D. thesis showed the equivalence between two definite integrals of arbitrary orders given by Holmgren and Riesz [4].

**Convergent Integrals**

The generalization of the integral $\int_a^b f(x)\, dx$ where [a,b] is finite interval and f(x) is continuous function in this interval or has a finite number of discontinuities of the first kind to cover the case when f(x) has an arbitrary set of discontinuous points while remaining bounded leads to Riemann integral.

If the interval [a,b] is infinite or f(x) is unbounded in a neighbourhood of a finite number of points then these integrals are called improper.

The name of improper integral of a function f(x) in a semi-open interval $[a,\infty]$ is defined as

$$\int_a^\infty f(x)\,dx = \lim_{l\to\infty}\int_a^l f(x)\,dx$$

If this limit exists then the integral is called convergent otherwise divergent or the integral does not exist. In this case we notice two possibilities:

1- $\lim_{l\to\infty}\int_a^l f(x)\,dx$ equals $\infty$ or $-\infty$

2- $\int_a^l f(x)\,dx$ does not tend to a limit whether finite or infinite. For example:-

$$\int_o^\infty \frac{dx}{x} = \infty \quad \text{(divergent)}$$

$\int_o^\infty \sin(x)\,dx = \lim_{l\to\infty}(1-\cos l)$ does not exist, in fact it oscillates between 0 and 2.

Geometrically this improper integral expresses the area between a curve and it’s asymptote [3,9].

An improper integral is called absolutely convergent if it converges after replacing f(x) by |f(x)|. But if it is convergent but not absolutely convergent then it is called conditionally convergent.

**<u>Tests of Convergence [3,9]:</u>**

(1) Cauchy test

$\int_{a}^{\infty} f(x)\,dx$ is convergent off $\forall \varepsilon > 0 \ \exists \mathbf{l}$ s.t $\forall q > \rho > \mathbf{l}$ then

$$\left| \int_{p}^{q} f(x)\,dx \right| < \varepsilon$$

(2) Suppose that for all values $x > N \geq a$ the inequality $|f(x)| \leq |\varphi(x)|$ holds. If $\int_{o}^{\infty} f(x)\,dx$ convergence absolutely then so is $\int_{a}^{\infty} f(x)\,dx$ .

**Note**:- $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{\mathbf{l} \to \infty \\ m \to -\infty}} \int_{m}^{\mathbf{l}} f(x)\,dx$

Where $\mathbf{l} \to \infty$ and $m \to -\infty$ are independently of each other.

**The Main Result**

Now we show the convergence of Riesz integral by compairing the integrand $|x - t|^n \ f(t)$ with certain functions.

**Theorem (1)**:- If $G(x,t) = |x - t|^n f(t) \leq \dfrac{1}{1 + t^2}$ then $f_n(x) = \int_{-\infty}^{\infty} G(x,t)\,dt$ converges absolutely.

**Proof:-**

$$|f_n(x)| \leq \int_{-\infty}^{\infty} |G(x,t)|\,dt \leq \int_{-\infty}^{\infty} \frac{dt}{1 + t^2} = \tan^{-1} t \Big)_{-\infty}^{\infty} = \pi$$

$\therefore f_n(x)$ convergence absolutely.

**Theorem (2)**:- If $G(x,t)=|x-t|^{n} f(t)\leq \dfrac{\cos t}{(t^2+a^2)(t^2+b^2)}$ then $f_n(x)$ is convergent.

**Proof:-**

$$f_n(x)=\int_{-\infty}^{\infty}|G(x,t)|\,dt\leq I_1=\int_{-\infty}^{\infty}\frac{\cos t}{(t^2+a^2)(t^2+b^2)}dt$$

Now we calculate $I_1$

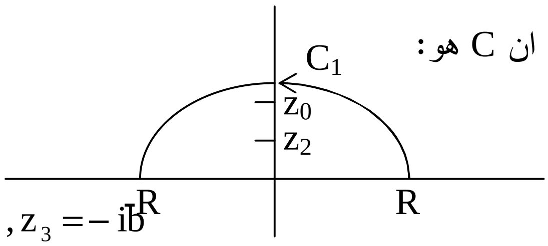
$$\int_{-\infty}^{\infty}\frac{\cos t\;dt}{(t^2+a^2)(t^2+b^2)}=\frac{\pi}{(a^2+b^2)}\left[\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right],\ a>b>0$$

$$\cos t=\operatorname{Re} e^{it}$$

$$\therefore\ I_1=\int_{-\infty}^{\infty}\frac{\cos t\;dt}{(t^2+a^2)(t^2+b^2)}=\operatorname{Re}\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{it}\;dt}{(t^2+a^2)(t^2+b^2)}$$

$$\int_C\frac{e^{iz}\;dz}{(z^2+a^2)(z^2+b^2)}=2\pi i\,(b_{1,0}+b_{1,2})$$

حيث ان C هو:




$$z_0=ia\,,\ z_1=-ia\,,\ z_2=ib\,,\ z_3=-ib$$

$$\because\ f(z)=\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$$

$$=\frac{e^{iz}}{(z-z_0)(z-z_1)(z-z_2)(z-z_3)}$$

$$\mathbf{Q}\,\varphi(z)=\left(z-z_0\right)f(z)=\frac{e^{iz}}{\left(z-z_1\right)\left(z-z_2\right)\left(z-z_3\right)}\qquad ,z\neq z_0$$

$$\therefore b_{1,0}=\lim_{z\to z_0}\varphi(z)=\frac{e^{-a}}{\left(z-z_1\right)\left(z-z_2\right)\left(z-z_3\right)}$$

$$\therefore b_{1,0}=\lim_{z\to z_0}\varphi(z)=\frac{e^{-a}}{\left(ia+ia\right)\left(-a^2+b^2\right)}$$

$$=\frac{e^{-a}}{2ia\left(b^2-a^2\right)}\frac{-2ia\,(b^2-a^2)}{-2ia\,(b^2-a^2)}=\frac{-2ia\,e^{-a}\,(b^2-a^2)}{4a^2\,(b^2-a^2)^2}$$

$$=\frac{-i\,e^{-a}}{2a\,(b^2-a^2)}$$

$$\varphi(z)=\left(z-z_2\right)f(z)=\frac{e^{iz}}{\left(z-z_0\right)\left(z-z_1\right)\left(z-z_3\right)}\qquad ,z\neq z_2$$

$$\therefore b_{1,2}=\lim_{z\to z_2}\varphi(z)=\frac{e^{-b}}{\left(-b^2+a^2\right)\left(ib+ib\right)}=\frac{e^{-b}}{2ib\,(a^2-b^2)}\frac{-2ib\,(a^2-b^2)}{-2ib\,(a^2-b^2)}$$

$$=\frac{-2ib\,e^{-b}\left(a^2-b^2\right)}{4b^2\left(a^2-b^2\right)^2}=\frac{-i\,e^{-b}}{2b\,(a^2-b^2)}$$

$$\lim_{R\to\infty}\;\int_{C_1\;R}^{-R}\frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)}=0$$

$$\left|\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\right| = \frac{e^{-y}}{|z^2+a^2|\,|z^2+b^2|} \le \frac{e^{-y}}{(R^2-a^2)(R^2-b^2)}$$

$$\le \frac{1}{(R^2-a^2)(R^2-b^2)} \quad , y \ge 0$$

$$\therefore \left|\int_{\substack{R \\ C_1}}^{-R} \frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)}\right| \le \int_R^{-R} \frac{|dz|}{(R^2-a^2)(R^2-b^2)} = \frac{\pi R}{(R^2-a^2)}$$

$$\therefore \lim_{R\to\infty} \int_{C_1\ R}^{-R} \frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)} = 0$$

$$\therefore \int_C \frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)} = \int_{-R}^{R} \frac{e^{it}\,dt}{(t^2+a^2)(t^2+b^2)} + \int_R^{-R} \frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)}$$

$$= 2\pi i\left(b_{1,0} + b_{1,2}\right)$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{it}\,dt}{(t^2+a^2)(t^2+b^2)} = 2\pi i\left(\frac{-i\,e^{-b}}{2b(a^2-b^2)} - \frac{C_1 - i\,e^{-a}}{2a(b^2-a^2)}\right)$$

$$= \pi\left(\frac{e^{-b}}{b(a^2-b^2)} + \frac{e^{-a}}{a(b^2-a^2)}\right)$$

$$= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\therefore\ I_1=\int_{-\infty}^{\infty}\frac{\cos t\ dt}{\left(t^2+a^2\right)\left(t^2+b^2\right)}=\operatorname{Re}\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{it}\ dt}{\left(t^2+a^2\right)\left(t^2+b^2\right)}$$

$$=\frac{\pi}{\left(a^2-b^2\right)}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$$

**Theorem (3)**:- If $G(x,t)=|x-t|^n f(t)\le\dfrac{\cos t}{(t+a)^2+b^2}$ , $b>0$ then $f_n(x)$ is convergent.

**Proof:-**

$$f_n(x)=\int_{-\infty}^{\infty}G(x,t)\,dt\le I_2=\int_{-\infty}^{\infty}\frac{\cos t}{(t+a)^2+b^2}\,dt$$

Now we calculate $I_2$

$$\cos t=\operatorname{Re}\left(e^{it}\right)$$

$$\therefore\ I_2=\int_{-\infty}^{\infty}\frac{\cos t\ dt}{(t+a)^2+b^2}=\operatorname{Re}\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{it}\ dt}{(t+a)^2+b^2}$$

$$\int_C\frac{e^{iz}}{(z+a)^2+b^2}=2\pi i\ \ b_{1,0}$$

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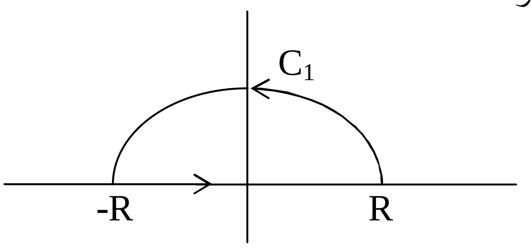

$$\left(z+a\right)^2+b^2=0$$

$$z^2+2az+\left(a^2+b^2\right)=0$$

$$z_0=\frac{-2a+\sqrt{4a^2-4\left(a^2+b^2\right)}}{2}$$

$$=\frac{-2a+\sqrt{4a^2-4a^2-4b^2}}{2}$$

$$=\frac{-2a+\sqrt{-4b^2}}{2}$$

$$=-a+ib$$

$$z_1=\frac{-2a+\sqrt{4a^2-4\left(a^2-b^2\right)}}{2}$$

$$=-a-ib$$

$$f(z)=\frac{e^{iz}}{(z+a)^2+b^2}=\frac{e^{iz}}{(z-z_0)(z-z_1)}$$

$$=\frac{e^{iz}}{\left(z-\left(-a+ib\right)\right)\left(z-\left(-a-ib\right)\right)}$$

$$\therefore \varphi(z)=\left(z-z_0\right)f(z)=\frac{e^{iz}}{\left(z-\left(-a-ib\right)\right)}\quad ,\ z\neq z_0$$

$$b_{1,0}=\lim_{z\to z_0}\varphi(z)=\frac{e^{i(-a+ib)}}{((-a+ib-(-a-ib))}=\frac{e^{-b}e^{-ia}}{(-a+ib+a+ib)}=\frac{e^{-b}e^{-ia}}{2ib}\frac{-2ib}{-2ib}$$

$$= \frac{-i2b\, e^{-b}\, e^{-ia}}{4b^{2}} = \frac{-i\, e^{-b}(\cos a - i\sin a)}{2b^{2}} = \frac{-i}{2b}\left(\frac{\cos a}{e^{b}} - i\frac{\sin a}{e^{b}}\right)$$

To prove

$$\lim_{R\to\infty} \int_{C_1}\int_{R}^{-R} \frac{e^{iz}\, dz}{(z+a)^{2}+b^{2}} = 0$$

$$\left|\frac{e^{iz}}{(z+a)^{2}+b^{2}}\right| = \left|\frac{e^{iz}}{(z-z_0)(z-z_1)}\right| = \frac{e^{-y}}{|z-z_0||z-z_1|} \le \frac{e^{-y}}{(R-z_0)(R-z_1)}$$

$$\le \frac{1}{(R-z_0)(R-z_1)}\ ,\ y\ge 0$$

$$\therefore \left|\int_{R}^{-R}\frac{e^{iz}}{(z+a)^{2}\ b^{2}}\, dz\right|_{C_1} \le \int_{R}^{-R}\frac{|dz|}{(R-z_0)(R-z_1)} = \frac{\pi R}{(R-z_0)(R-z_1)}$$

$$\therefore \lim_{R\to\infty} \int_{C_1}\int_{R}^{-R} \frac{e^{iz}\, dz}{(z+a)^{2}+b^{2}} = 0$$

$$\therefore \int_{C} \frac{e^{iz}\, dz}{(z+a)^{2}+b^{2}} = \int_{-R}^{R}\frac{e^{it}\, dt}{(t+a)^{2}+b^{2}} + \int_{R}^{-R}\frac{e^{iz}\, dz}{(z+a)^{2}+b^{2}} = 2\pi i\, b_{1,0}$$

$$= 2\pi i\left(\frac{-i}{2b}\left(\frac{\cos a}{e^{b}} - \frac{\sin a}{e^{b}}\right)\right)$$

$$=\frac{\pi\cos a}{b\ e^{b}}-i\ \frac{\pi\sin a}{b\ e^{b}}$$

$$\therefore\ \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{it}\ dt}{(t+a)^2+b^2}=\frac{\pi\cos a}{b\ e^{b}}-i\frac{\pi\sin a}{b\ e^{b}}$$

$$\therefore\ I_2=\int_{-\infty}^{\infty}\frac{\cos t\ dt}{(t+a)^2+b^2}=\operatorname{Re}\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{it}\ dt}{(t+a)^2+b^2}\ =\frac{\pi\cos a}{b\ e^{b}}$$

$$\therefore\ \int_{-\infty}^{\infty}\frac{\cos t\ dt}{(t+a)^2+b^2}=\frac{\pi\cos a}{b\ e^{b}}$$

**Theorem (4)**:- If $G(x,t)=|x-t|^{n}f(t)\le\frac{\sin t}{t^2+4t+5}$ then $f_n(x)$ is convergent.

**Proof:-**

$$f_n(x)=\int_{-\infty}^{\infty}G(x,t)\,dt\le I_3=\int_{-\infty}^{\infty}\frac{\sin t\,dt}{t^2+4t+5}$$

Now we calculate $I_3$

$$\therefore \sin t=\operatorname{Ln}\left(e^{it}\right)$$

$$\therefore\ I_3=\int_{-\infty}^{\infty}\frac{\sin t}{t^2+4t+5}dt=\operatorname{Ln}\ \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{it}}{t^2+4t+5}dt$$

$$\int_{C}\frac{e^{iz}}{z^2+4z+5}dz=2\pi i\ \ b_{1,0}$$

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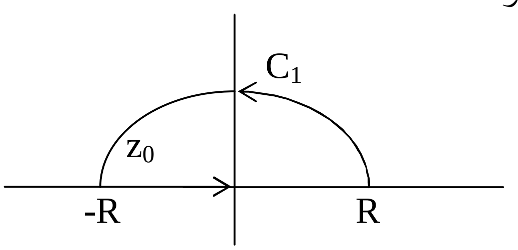

$$z^2 + 4z + 5 = 0$$

$$z_0 = \frac{-4 + \sqrt{16 - 20}}{2} = \frac{-4 + \sqrt{-4}}{2} = -2 + i$$

$$z_1 = \frac{-4 - \sqrt{16 - 20}}{2} = \frac{-4 - \sqrt{-4}}{2} = -2 - i$$

$$f(z) = \frac{e^{iz}}{z^2 + 4z + 5} = \frac{e^{iz}}{(z - z_0)(z - z_1)} = \frac{e^{iz}}{\left(z - (-2 + i)\right)\left(z - (-2 - i)\right)}$$

$$\therefore \varphi(z) = \left(z - z_0\right) f(z) = \frac{e^{iz}}{\left(z - \left(-2 - i\right)\right)} \quad , \; z \neq z_0$$

$$\therefore b_{1,0} = \lim_{z \to z_2} \varphi(z) = \frac{e^{i(-2+i)}}{\left(-2 + i + 2 + i\right)} = \frac{e^{-2i} \; e^{-1}}{2i} = \frac{e^{-2i} \; e^{-1}}{2i} \; \frac{-2i}{-2i} = \frac{-2i \; e^{-2i} \; e^{-1}}{4}$$

$$= \frac{-e^{-1} i \left(\cos_2 - i \sin_2\right)}{2} = \frac{i}{2}\left(-\frac{\cos 2}{e} + i \frac{\sin 2}{e}\right)$$

To prove

$$\lim_{R \to \infty} \int_{C_1 \; R}^{-R} \frac{e^{iz}}{z^2 + 4z + 5} dz = 0$$

$$\left|\frac{e^{iz}}{z^2 + 4z + 2}\right| = \left|\frac{e^{iz}}{(z - z_0)(z - z_1)}\right| = \frac{e^{-y}}{\left|z - z_0\right|\left|z - z_1\right|} \leq \frac{e^{-y}}{(R - z_0)(R - z_1)}$$

$$\leq \frac{1}{(R-z_0)(R-z_1)} \quad , y \geq 0$$

$$\therefore \left| \int_{\substack{R \\ C_1}}^{-R} \frac{e^{iz}}{z^2+4z+5} dz \right| \leq \int_{\substack{R \\ C_1}}^{-R} \frac{1}{(R-z_0)(R-z_1)} |dz| = \frac{\pi R}{(R-z_0)(R-z_1)}$$

$$\therefore \lim_{R\to\infty} \int_{\substack{R \\ C_1}}^{-R} \frac{e^{iz}}{z^2+4z+5} dz = 0$$

$$\therefore \int_C \frac{e^{iz}\, dz}{z^2+4z+5} = \int_{-R}^{R} \frac{e^{it}\, dt}{t^2+4t+5} + \int_{\substack{R \\ C_1}}^{-R} \frac{e^{iz}\, dz}{z^2+4z+5} = 2\pi i\, b_{1,0}$$

$$= 2\pi i \left( \frac{i}{2} \left( -\frac{\cos 2}{e} + i \frac{\sin 2}{e} \right) \right)$$

$$= -\pi \left( -\frac{\cos 2}{e} + i \frac{\sin 2}{e} \right) = \pi \frac{\cos 2}{e} - i\pi \frac{\sin 2}{e}$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{it}\, dt}{t^2+4t+5} = \pi \frac{\cos 2}{e} - i\pi \frac{\sin 2}{e}$$

$$\therefore I_3 = \int_{-\infty}^{\infty} \frac{\sin t\ dt}{t^2+4t+5} = \text{Ln} \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{it}\, dt}{t^2+4t+5} = -\pi \frac{\sin 2}{e}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin t\ dt}{t^2+4t+5} = \frac{-\pi}{e} \sin 2$$

**Lemma [8]**:- We can simply prove that:

1. $\int_{-\infty}^{\infty} \frac{2t^2 - 1}{t^4 + 5t^2 + 4} dt = \frac{\pi}{2}$

2. $\int_{-\infty}^{\infty} \frac{\cos t}{(t^2 + 1)^2} dt = \frac{\pi}{e}$

3. $\int_{-\infty}^{\infty} \frac{t \sin a t}{t^4 + 4} dt = \frac{\pi}{2} e^{-a} \sin a \qquad a > 0$

4. $\int_{-\infty}^{\infty} \frac{t\,dt}{(t^2 + 1)(t^2 + 2t + 2)} = \frac{-\pi}{5}$

5. $\int_{-\infty}^{\infty} e^{-t^2} \cos(2bt)\,dt = \sqrt{\pi}\, e^{-b^2}$

therefore if $G(x, t) \leq$ the integrand of the above then $f_n$ is convergent.

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**<u>المستخلص</u>**: تمت دراسة تقارب التكامل الكسري المعرف من قبل Riesz وذلك بمقارنة المقدار داخل التكامل $f(t)\ |x-t|^n$ حيث $0<n<1$ مع بعض الدوال. استخدمنا مبدأ تغيير الاحداثيات ومفاهيم التحليل العقدي لبرهنة التقارب. وهذا يمكننا من استخدام طريقة اختبار المقارنة بالنسبة للتقارب بصورة أوسع.