Quasi- essential submodules

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المستخلص
لتكن R حلقة ابدالية بمحايد و M مقاساً أحادياً على R. المقاس الجزئي الفعلي N من القياس M يدعى مقاس جزئي جوهري إذا كان NOK ≠ (o) لكل مقاس جزئي غير صفري K من M المقاس. الجزئي الفعليL من المقاس M يدعى مقاس جزئي شبه جوهري إذا كان LOP ≠(O) لكل مقاس جزئي أولي غير صفريP منM وهو تعميم إلى المقاس الجزئي الجوهري.
في هذا البحث أعطينا تعميم آخر للمقاسات الجزئية الجوهرية وأسميناها المقاسات الجوهرية ظاهرياً . حيث عرفنا المقاس الجزئي الفعلي الجوهري ظاهرياً H من المقاسM ، إذا كان HOQ ≠ (O) لكل مقاس جزئي أولي ظاهرياً Q من M . كل مقاس جزئي جوهري يعطي مقاس جزئي جوهري ظاهرياً (شبه جوهري) وكل مقاس جزئي جوهري ظاهرياً يكون مقاساً جزئياً شبه جوهري والعكس غير صحيح . هدفنا في هذا البحث هو دراسة الخواص الأساسية للمقاسات الجزئية الجوهرية ظاهرياً والمثاليات شبه الجوهرية في R . ودرسنا المقاسات التي تحقق Acc (Dcc) للمقاسات الجزئية الجوهرية ظاهرياً.

ABSTRACT

Let R be a commutative ring with identity, and M be a unitary R-module. A proper submodule N of an R-module M is called an essential if $N \cap K \neq (0)$ for each non-zero submodule K of M, and a proper submodule L of an R-module M is called semi-essential if $L \cap P \neq (0)$ for each non-zero prime submodule P of M, which is a generalization of essential submodules.

In this paper we give another generalization of essential submodule, we call it a quasi-essential submodulei, where we call a proper submodule H of an R-module M a quasi-essential if $H \cap Q \neq (0)$ for each non-zero quasi-prime R-submodule Q of M . Every essential submodule is a quasi-essential submodule

(semi-essential submodule), and every quasi-essential submodule is semi-essential but the converse is not true. Our main goal in this work is to study the basic properties of quasi-essential submodules, and semi-essential ideals in R. Also we study the modules that satisfies Acc(Dcc) on quasi-essential submodules.

Introduction

Let R be a commutative ring with identity, and M be a unitary R-module.

A proper submodule N of an R-module M is called an essential if $N \cap K \neq (0)$ for each non-zero submodule K of M [4] .and a proper submodule L of an R-module M is called semi-essential if $L \cap P \neq (0)$ for each non-zero prime submodule P of M,[2] which is a generalization of essential submodules. We give another generalization of essential submodule, call it quasi-essential submodule. A proper submodule N' of M is called quasi-essential submodule if $N' \cap Q \neq (0)$ for each non-zero quasi-prime submodule Q of M , where quasi-prime submodule was introduced in[1] ,as a generalization of prime submodule which was introduced in [6],recall that an R-submodule N of an R-module M is called a prime submodule, if $rm \in M, m \in M, r \in R$, then either $m \in N \ or \ r \in [N:M]$, where $[N:M] = \{r \in R : rM \subseteq N\}$, and we recall that an R-submodule Q of R-module M is a quasi-prime, if $r_1 r_2 m \in Q, m \in M, r_1, r_2 \in R$ then either $r_1 m \in Q \ or \ r_2 m \in Q$. Since every prime submodule is a quasi-prime and the converse is not true [1].

We prove that every quasi-essential submodule is semi-essential but the converse is not true see Example and remark1.2 .

In the first section of this paper, we introduce the concept of quasi-essential submodule, and study some basic properties of this concept.

In the second section, we introduce the concept of a quasi-essential homomorphism. In the third section, we study a quasi- submodules in multiplication module. In the fourth section we study modules that satisfies Acc(Dcc) on quasi-essential submodules.

$1 Quasi-essential submodules

In this section, we introduce the concept of a quasi-essential submodule as a generalization of essential submodule and we give the basic properties characterization, and examples of this concept.

Definition 1.1

A non-zero submodule N of an R-module M is called a quasi-essential submodule if $N \cap Q \neq (0)$ for each non-zero quasi-prime submodule Q of M.

Examples and Remarks 1.2

1- Every essential submodule is a quasi-essential submodule, but the converse is not true as the following example says: In the Z-module Z_{12}, the submodule $(\bar{6})$ is a quasi-essential submodule but not essential submodule of Z_{12}, since $(\bar{6}) \cap (\bar{4}) = (\bar{0})$. *But* $(\bar{6}) \cap (\bar{3}) \neq (\bar{0})$ *and* $(\bar{6}) \cap (\bar{2}) \neq (\bar{0})$ where $(\bar{2})$ *and* $(\bar{3})$ are the only quasi-prime submodule of Z_{12}.

2- Every quasi-essential submodule is semi-essential , but the converse is not true as the following example says: In the Z-module $M = Z \oplus Z$, the only prime submodules are of the forms $Z \oplus pZ$ *and* $pZ \oplus Z$ where p is a prime number. The submodule $N = 0 \oplus Z$ of M is semi-essential, but not a quasi-essential, since $(2Z \oplus 0) \cap (0 \oplus Z) = (0)$ where $2Z \oplus 0$ is quasi-prime submodule of M not prime submodule [1] .

3- A submodule of quasi-essential submodule need not be quasi-essential submodules. In the Z-module Z_{12} the submodule $(\bar{2})$ is a quasi-essential submodule of Z_{12} , but the submodule $(\bar{4}) \subseteq (\bar{2})$ is not a quasi-essential submodule of Z_{12}, since $(\bar{4}) \cap (\bar{3}) = (\bar{0})$.

4- Every submodule of the Z-module Z is quasi-essential submodule.

5- Every proper submodule of the Z-module Z_{p^∞} is quasi-essential submodules.

6- In semi-simple R-module M, the only quasi-essential submodule is M itself.

Theorem1.3

Let M be an R-module, and N_1,N_2 are submodules of M such that $N_1 \subseteq N_2$. If N_1 is a quasi-essential submodule of M, then N_2 is quasi-essential submodule of M.

Proof: Suppose that for some quasi-prime submodule Q of M, $N_2 \cap Q = (0)$. But N_1 is subset of N_2, then $N_1 \cap Q = (0)$. Since N_1 is a quasi-essential submodule of M, then $Q = (0)$. Hence N_2 is a quasi-essential submodule of M.

As a direct consequence of the above theorem we get the following corollaries:

Corollary1.4

Let M be an R-module, and N_1,N_2 be submodules of M with $N_1 \cap N_2$ is a quasi-essential submodule of M, then N_1 and N_2 are quasi-essential submodule of M.

The converse of corollary 1.4 is not true in general, for example, in the Z-module Z_{36}, the submodules $(\overline{12})$ and $(\overline{18})$ are quasi-essential submodule of Z_{36}. But $(\overline{12}) \cap (\overline{18}) = (0)$ is not a quasi-essential submodule of Z_{36}, since the only quasi-prime submodule of Z_{36} are $(\overline{2})$ *and* $(\overline{3})$.

Now we gives a partial converse of corollary 1.4.

Proposition1.5

Let M be an R-module, and N_1, N_2 be two submodules of M with N_2 dose not contained in any quasi-prime submodule of M. If N_1 is a quasi-essential submodule of N_2 and N_2 is quasi-essential submodule of M. Then $N_1 \cap N_2$ is a quasi-essential submodule of M.

Proof Suppose that $(N_1 \cap N_2) \cap Q = (0)$ for some quasi-prime submodule Q of M. Since $N_2 \not\subset Q$, then $(N_2 \cap Q)$ is quasi-prime submodule of N_2 by[1,prop.2.1.12]. But N_1 is a quasi-essential submodule of N_2, then $N_2 \cap Q = (0)$. But N_2 is a quasi-essential submodule of M, then $Q=(0)$ Hence $N_1 \cap N_2$ is a quasi-essential submodule of M.

Proposition1.6

Let M be an R-module, and N_1, N_2 be two submodules of M such that N_1 is essential submodule of M and N_2 is quasi-essential submodule of M. Then $N_1 \cap N_2$ is a quasi-essential submodule of M.

proof

Let Q be a non-zero quasi-prime submodule of M. Since N_2 is quasi-essential submodule of M, then $N_2 \cap Q \neq (0)$. Since N_1 is an essential submodule of M , then $N_1 \cap (N_2 \cap Q) \neq (0)$, and so we get $(N_1 \cap N_2) \cap Q \neq (0)$,which implies that $N_1 \cap N_2$ is a quasi-essential submodule of M.

Before we give of consequence of theorem 1.3 we recall the following definitions :

Let N be a submodule of an R-module M, and S be a multiplicative set of R. $N(S)=\{x \in M : \exists t \in S, tx \in N\}$, $N(S)$ is an R-submodule of M contains N [5]. We define a closure of a submodule N, denoted by $cl(N)=\{m \in M : [N:(m)] \text{ essential in } R\}$. $cl(N)$ is a submodule of M , and $N \subseteq cl(N)$.

The definition of M-radical of a submodule N of an R-module was given in [5] as the intersection of all prime submodule of M containing N, and denoted by $\sqrt{N}$.

Corollary 1.7

Let M be an R-module, and N is quasi-essential submodule of M. Then

1. $N(S)$ is a quasi-essential submodule of M.
2. $cl(N)$ is a quasi-essential submodule of M.
3. $\sqrt{N}$ is a quasi-essential submodule of M.

Corollary 1.8

Let M be an R-module , and N_1, N_2 are two submodules of M such that either N_1 or N_2 is quasi-essential submodule of M, then $N_1 + N_2$ is a quasi-essential submodule of M.

Theorem1.9

Let N be a submodule of M. Then N is a quasi-essential submodule of an R-module M if and only if $\left[N \underset{M}{:} I\right]$ is a quasi-essential submodule of M for each non-zero ideal I of R.

Proof

Suppose that N is a quasi-essential submodule of M, and I is an ideal of R. Since $N \subseteq \left[N \underset{M}{:} I\right]$, then by theorem 1.3 $\left[N \underset{M}{:} I\right]$ is a quasi-essential submodule of M.

The converse followed by taking I=R.

Proposition 1.10

Let M be an R-module, then a non-zero R-submodule N of M is a quasi-essential submodule of M if and only if for each non-zero a quasi-prime submodule Q of M, there exist x in Q and there exist a non-zero r in R such that

$0 \neq rx \in N$.

Proof

Suppose that N is a quasi-essential submodule of M, then $N \cap Q \neq (0)$ for each non-zero a quasi-prime submodule Q of M. Then $\exists x \neq 0, x \in N \cap Q$. Thus x∈N and x∈Q. Hence $0 \neq 1x \in N$.

Conversely: Suppose that for each non-zero quasi-prime submodule Q' of M, $\exists x \neq 0, x \in Q'$ *and* $\exists r \neq 0, r \in R \ni rx \in N$,since $r \in R$ *and* $x \in Q'$,then $rx \in Q'$ so that $0 \neq rx \in N \cap Q'$. Thus $N \cap Q' \neq (0)$ and hence N is a quasi-essential submodule of M.

Proposition 1.11

Let M_1 and M_2 be two R-modules, and let $M = M_1 \oplus M_2$, such that every submodule N of M is of the form $N = N_1 \oplus N_2$, where N_1 and N_2 are submodules of M_1 and M_2 respectively, if N_1 is a quasi-essential submodule of M_1 and N_2 is a quasi-essential submodule of M_2, then $N = N_1 \oplus N_2$ is a quasi-essential submodule of M.

Proof

Let Q be a quasi-prime submodule of M , then $Q = Q_1 \oplus Q_2$ where $Q_1$is a quasi-prime submodule of M_1 and $Q_2$is a quasi-prime submodule of M_2 [1,prop.2.2.7].

Since N_1 is a quasi-essential submodule of M_1, then by prop1.10 $\exists 0 \neq x_1 \in Q_1 \;\&\; 0 \neq r \in R \ni 0 \neq rx_1 \in N_1$ and Since N_2 is a quasi-essential submodule of M_2, then by prop1.10 $\exists 0 \neq x_2 \in Q_2 \;\&\; 0 \neq r \in R \ni 0 \neq rx_2 \in N_2$.

Hence $(0,0) \neq (rx_1, rx_2) \in N = N_1 \oplus N_2$. i.e. $(0,0) \neq r(x_1, x_2) \in N_1 \oplus N_2$ where $(0,0) \neq (x_1, x_2) \in Q$ .That is N is a quasi-essential submodule of M.

Corollary 1.12

Let M be an R-module, and N be a submodule of M. If N is a quasi-essential submodule of M, then $N^2 = N \oplus N$ is a quasi-essential submodule of $M^2 = M \oplus M$.

Proposition 1.13

Let $M = M_1 \oplus M_2$ be the direct sum of two R-modules M_1 and M_2. If N_1 is a quasi-essential submodule of M_1 ,then $N_1 \oplus M_2$ is a quasi-essential submodule of M.

Proof

Since N_1 is a quasi-essential submodule of M_1 and M_2 is a quasi-essential submodule of M_2, then by prop. 1.11 ,we have $N_1 \oplus M_2$ is a quasi-essential submodule of M.

In the following proposition, we study the behavior of a quasi-essential submodules under localization.

Proposition 1.14

Let M be an R-module, and N is a submodule of M. If N_S is a quasi-essential R_S-submodule of M_S , then N is a quasi-essential submodule of M.

Proof

Let Q be a quasi-prime submodule of M, then Q_S is a quasi-prime submodule of M_S by [1,prop.2.3.3]. Since N_S is quasi-essential R_S-submodule of M_S , then $N_S \cap Q_S \neq (0)$. That is $(N \cap Q)_S \neq (0)$, then $(N \cap Q) \neq (0)$, which is implies that N is a quasi-essential submodule of M.

$2:Quasi-essential Homomorphism

In this section we introduce the concept of quasi- essential homomorphism, and study properties of this concept.

Definition 2.1

An R-homomorphism $f:M \to M'$, where M&M' are two R-modules, is called a quasi- essential homomorphism if f(M) is a quasi-essential submodule of M'.

The proof of the following proposition is straightforward and hence omitted.

Proposition 2.2

Let M be an R-module, and N be an R-submodule of M, then N is a quasi-essential submodule of M if and only if the inclusion function $i:N \to M$ is a quasi- essential monomorphism.

Proposition 2.3

Let M & M' be an R-modules and $f:M \to M'$ be an R-epimorphism, then

1. If N is a quasi-essential submodule of M, then f(N) is a quasi-essential submodule of M'.
2. If N' is a quasi-essential submodule of M' and $Kerf \subseteq Q$ for each quasi- prime submodule Q of M, then $f^{-1}(N')$ is a quasi-essential submodule of M .

Proof

1- Let Q' be a quasi- prime submodule of M', then $f^{-1}(Q')$ is quasi- prime submodule of M [1,prop.2.3.1] . But N is a quasi-essential submodule of M, then $N \cap f^{-1}(Q') \neq (0) \Rightarrow f(N) \cap Q' \neq (0)$.

2- Suppose that $f^{-1}(N') \cap Q = (0)$ for some quasi- prime submodule Q of M. Thus $N' \cap f(Q) = (0)$ since Q is a quasi-prime submodule of M , with $Kerf \subseteq Q$, then f(Q) is a quasi-prime submodule of M' by[1,prop.2.3.1] . But N' a quasi-essential submodule of M', then $f(Q) = (0)$ which implies that $Q \subseteq \ker f \subseteq f^{-1}(N')$ and hence $Q = f^{-1}(N') \cap Q = (0)$ that is Q=(0). Hence $f^{-1}(N')$ is a quasi-essential submodule of M.

Corollary2.4

Let K and N be two submodule of an R-module M and $K \subseteq N$ & $N \subseteq Q$ for each quasi- prime submodule Q of M. Then $\frac{N}{K}$ is a quasi-essential submodule of $\frac{M}{N}$ if and only if N is a quasi-essential submodule of M.

Proposition 2.5

Let M_1 and M_2 be two R-modules, and let $Hom_R(M_1,K)$ be a proper sub -module of $Hom_R(M_1,M_2)$ for any submodule K of M. If $Hom_R(M_1,N)$ is a quasi-essential submodule of $Hom_R(M_1,M_2)$, then N is a quasi-essential submodule of M_2.

Proof

Let Q be a quasi- prime submodule of M_2, then $Hom_R(M_1,Q)$ is a quasi- prime submodule of $Hom_R(M_1,M_2)$ by [1,prop.2.3.6]. But $Hom_R(M_1,N)$ is a quasi-essential submodule of $Hom_R(M_1,M_2)$, then $\exists f\,, 0 \neq f \in Hom_R(M_1,Q) \& 0 \neq r \in R \ni 0 \neq rf \in Hom_R(M,N)$ [by prop. 1.10]. i.e. $rf(m) \in N\ \forall m \in M_1 \& 0 \neq f(m) \in Q$. Therefore N is a quasi-essential submodule of M_2.

Corollary2.6

If $Hom_R(M,N)$ is a quasi-essential submodule of $Hom_R(M,M)$, then N is a quasi-essential submodule of M.

$3: Quasi –essential submodules in multiplication modules

This section is devoted to study quasi-essential submodules in multiplication modules.

We need to recall the following definitions:

Recall that an R-module M is called multiplication module if for each submodule N of M, there exist an ideal I of R such that $N = IM$ [3].

Recall that a non-zero ideal I of a ring R is semi-essential ideal if $I \cap P \neq (0)$ for every non-zero prime ideal P of R [2].

We start by the following proposition.

Proposition 3.1

Let M be faithful multiplication R-module, and N is a submodul-e of M such that $N = IM$ for some ideal I of R. Then N is a quasi-essential submodule of M if and only if I is semi-essential ideal of R.

Proof

Suppose that N a quasi-essential submodule of M, and let $I \cap P = (0)$ for some prime ideal P of R. Since M is faithful multiplication module, then $(0) = (I \cap P)M = IM \cap PM$. Since P is prime ideal , then PM is quasiprime submodule of M [3.lemma2.10]. which implies that PM is quasi-prime submodule of M [1.Remark 2.1.2(1)]. Since $N = IM$ is a quasi-essential submodule of M, then PM=(0). But M is faithful module, then P=(0). Therefore I is semi-essential ideal of R.

Conversely: Suppose that I is semi-essential ideal of, and let $N \cap Q = (0)$, for some quasi-prime submodule Q of M. Since M is multiplication , then Q is a prime submodule of M by [1,Prop.2.1.9]. Now since M is multiplication module, then there exist an ideal P of R such that Q=PM [3,Lemma2.11]. Hence $(0) = N \cap Q = IM \cap PM = (I \cap P)M$. But M is faithful multiplication, then $I \cap P = (0)$. Since I is semi-essential ideal of R, then P=(0). Hence Q=PM=(0). Therefore N is a quasi-essential submodule of M.

Proposition 3.2

Let M be faithful multiplication R-module, and N is a submodule of M. Then N is a quasi-essential submodule of M if and only if $[N:(x)]$ is semi-essential ideal of R for each x in M.

Proof

Suppose that N is a quasi-essential submodule of M. Then by prop.3.1 $[N:M]$ is semi-essential ideal of R. But for each x in M $[N:M] \subseteq [N:(x)]$. Since M is faithful multiplication , then $[N:M]M \subseteq [N:(x)]M$ [3]. But $[N:M]M$ is a quasi-essential submodule of M . Therefore $[N:(x)]$ is semi-essential ideal of R by Prop.3.1.

Conversely: Suppose that $[N:(x)]$ is semi-essential ideal of R for each x in M. Let Q be non-zero quasi-prime submodule of M such that $0 \neq x \in Q$. Since M is multiplication module, then Q is prime submodule of M by [1,Prop.2.1.9], and Q=PM, where P is a prime ideal of R. But $[N:(x)]$is semi-essential ideal of R, then $[N:(x)] \cap P \neq (0)$, and since M is faithful multiplication, then $[N:(x)]M \cap PM \neq (0)$. That is $N \cap Q \neq (0)$ and hence N is a quasi-essential submodule of M.

Theorem3.3

Let M be faithful multiplication R-module and N=IM is a submodule of M for some ideal I of R. Then the following statement are equivalents:

1. N is a quasi-essential submodule of M.
2. I is semi-essential ideal of R.
3. N is semi-essential submodule of M .
4. $[N:(x)]$is semi-essential ideal of R for all $0 \neq x \in M$.

Proof

$1 \rightarrow 2$ $[Prop.3.1]$.

$2 \rightarrow 3$ $[2, Prop\, 2.1.1]$.

$3 \rightarrow 4$ $[2, Prop\, 2.1.2]$.

$4 \rightarrow 1$ *by* Prop.3.2.

$4: Modules that satisfies Acc (Dcc) on Quasi –essential submodules

An R-module M is said to be satisfy the ascending (descending) chain condition Acc (Dcc) on a quasi-essential submodule of M if every ascending (descending) chain of a quasi-essential submodules $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ respectively$(N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots)$terminates.

We start with the following result:

Proposition 4.1

An R-module M satisfy Acc (Dcc) on a quasi-essential submodules of M if each a quasi-essential submodule of M is finitely generated.

Proof

Let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ be ascending) chain of a quasi-essential submodules N_i of M. Put $\sum_{i \in I} N_i = N$, then N is a quasi-essential submodule of M by theorem1.3, and hence N is finitely generated sub module of M. Therefore, there exist a finite set $I_0 \subseteq I$, such that $\sum_{i \in I_0} N_i = N$. Hence the chain $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ is terminates.

<u>Proposition 4.2</u>

Let M be an R-module such that M satisfy Acc (Dcc) on a quasi-essential submodule of M . Then $\frac{M}{N}$ satisfy Acc (Dcc) on a quasi-essential submodule of $\frac{M}{N}$ for each submodule N of M contained in every a quasi-prime submodule of M .

<u>Proof</u>

Suppose that M satisfy Acc (Dcc) on a quasi-essential submodules $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ of M. Let $\pi : M \to \frac{M}{N}$ is a natural homomorphism and let $\frac{N_1}{N} \subseteq \frac{N_2}{N} \subseteq \cdots \subseteq \frac{N_n}{N} \subseteq \cdots$ be ascending chain of a quasi-essential submodule of $\frac{M}{N}$ such that $N \subseteq N_i$ for each i. Hence $\pi^{-1}\left(\frac{N_i}{N}\right) = N_i$ is a quasi-essential submodules M for each i by Prop.2.3. Hence $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ is ascending chain of a quasi-essential submodule of M. But M satisfy Acc on a quasi-essential submodules of M, then there exist a positive integer n such that $N_n = N_{n+1} = \cdots$ and $\frac{N_n}{N} = \frac{N_{n+1}}{N} = \cdots$. Therefore $\frac{M}{N}$ satisfy Acc on a quasi-essential submodule of $\frac{M}{N}$.

Similarly for Dcc.

The next proposition gives the relation between the R-module M satisfy Acc (Dcc) on a quasi-essential submodules of M and a ring R that satisfy Acc (Dcc) on semi-essential ideal.

Proposition 4.3

Let M be finitely generated faithful multiplication R-module. Then M satisfy Acc (Dcc) on a quasi-essential submodule of M if and only if R satisfy Acc (Dcc) on semi-essential ideal.

Proof

Suppose that M satisfy Dcc on a quasi-essential submodule of M, and let

$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be descending chain of semi-essential ideal of R. Then $I_1M \supseteq I_2M \supseteq \cdots \supseteq I_nM \supseteq \cdots$ be a descending chain on a quasi-essential submodules of M by Prop.3.1. But M is satisfy Dcc on a quasi-essential submodules of M , then there exist a positive integer n such that $I_nM = I_{n+1}M = \cdots$.But M is finitely generated faithful multiplication R-module , then $I_n = I_{n+1} = \cdots$[3.Th.3.1]. Hence R satisfy Dcc on semi-essential ideals.

Conversely; Suppose that R satisfy Dcc on semi-essential ideals of R , and let

$N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be descending chain of a quasi-essential submodules of M. Since M is multiplication R-module, then

$N_i = I_iM$ for some semi-essential ideal I_i of R for each

i=1,2,…,n…byProp.3.1. Thus $I_1M \supseteq I_2M \supseteq \cdots \supseteq I_nM \supseteq \cdots$ and since M is finitely generated faithful multiplication R-module, then

$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be descending chain of semi-essential ideals of R by[3,Th3.1] . But R satisfy Dcc on semi-essential ideals, thus there exist a positive integer n such that $I_n = I_{n+1} = \cdots$.

Hence $I_nM = I_{n+1}M = \cdots$. Therefore M satisfy Dcc on quasi-essential submodules of M .

Similar proof for Acc.

Theorem 4.4

Let M be finitely generated faithful multiplication R-module. Then the following statement are equivalent:

1- M satisfy Acc (Dcc) on quasi-essential submodules of M.

2- R satisfy Acc (Dcc) on semi-essential ideals.

3- $S = \underset{R}{End}(M)$ satisfies Acc(Dcc) on semi-essential ideals.

4- M satisfy Acc (Dcc) on quasi-essential submodules of as S-module.
5-M satisfy Acc (Dcc) on semi-essential submodules as S-module.
6- M satisfy Acc (Dcc) on semi-essential submodules as R-module.

Proof

$1 \rightarrow 2 \quad [Prop.4.3]$

$2 \rightarrow 3$ Since M is finitely generated faithful multiplication R-module, then $R \cong S$ by [7,Cor3.3], so R satisfy Acc (Dcc) on semi-essential ideals if and only if S satisfies Acc(Dcc) on semi-essential ideals.

$3 \rightarrow 4$ [prop.3.1].

$4 \rightarrow 5$ [Th.3.3].

$5 \rightarrow 6$ Since M is finitely generated faithful multiplication R-module then $R \cong S$ by [Cor.3.3], then S satisfies Acc(Dcc) on semi-essential ideals so is R and hence M satisfy Acc (Dcc) on a quasi-essential submodules of M by Th.3.3.

$6 \rightarrow 1$[Th.3.3].

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